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Lagrangian flows and the one-dimensional Peano phenomenon for ODEs

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ABSTRACT

We consider the one-dimensional ordinary differential equation with a vector field which is merely continuous and nonnegative, and satisfies a condition on the amount of zeros. Although it is classically known that this problem lacks uniqueness of classical trajectories, we show that there is uniqueness for the so-called regular Lagrangian flow (by now usual notion of flow in nonsmooth situations), as well as uniqueness of distributional solutions for the associated continuity equation. The proof relies on a space reparametrization argument around the zeros of the vector field.

1. Introduction

1.1. The Peano phenomenon for the square root

Let us consider the vector field $f:\mathbb{R}\to\mathbb{R}$ given by the square root function: $f(x)=\sqrt{|x|}$. We are interested in the study of the one-dimensional ordinary differential equation

$$\dot\gamma(t)=f\big(\gamma(t)\big)=\sqrt{\big|\gamma(t)\big|}\quad \text{for } t\in[0,T] \tag{1.1}$$

with some prescribed initial datum $\gamma(0)=x_0\in\mathbb{R}$. Since f is not Lipschitz (due to the singularity at $x=0$), the Cauchy–Lipschitz theory (regarding existence and uniqueness of classical solutions to

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ODEs) does not apply, and in fact uniqueness of the trajectories γ fails. For instance, when $x_0 = 0$, it is easy to realize that (1.1) has the solution

$$\gamma^{t_0}(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant t_0, \\ \frac{1}{4}(t-t_0)^2 & \text{it } t_0 < t \leqslant T \end{cases}$$

for every value of the parameter $t_0 \in [0, T]$. This means that the solution can stay at rest in the point $x = 0$ for an arbitrary amount of time t_0, before exiting from it. This example is known as the Peano phenomenon. We notice that (while uniqueness does not hold in general) the continuity of f is enough to guarantee (local) existence of a solution, due to the so-called Peano theorem. We refer for instance to [14] for an introduction to the classical theory of ordinary differential equations and for further remarks on this example.

From a heuristic point of view, what goes wrong in this example is precisely the stopping of the trajectories at the point $x = 0$. If we consider initial data in the interval $[-1, 0]$, and for any such initial data we pick exactly the trajectory that stays indefinitely at $x = 0$ once it reaches this point, then we see that the whole interval $[-1, 0]$ collapses to one point in finite time, under the action of this flow.

However, we notice that the lack of uniqueness is due to the simultaneous occurrence, at the origin, of both the lack of regularity and the vanishing of f. Indeed, for general continuous vector fields f, under the assumption that $f(x_0) \neq 0$, we can divide $\dot\gamma = f(\gamma)$ by $f(\gamma)$ and integrate in a neighborhood of x_0. Thus γ must be locally given by an explicit formula involving the inverse of the primitive of $1/f$, and uniqueness follows. The effect of the zeros of the vector field on the lack of uniqueness is also one of the main points in the analysis of two-dimensional flows in [1]: in that paper, the uniqueness is characterized via a property (the so-called "weak Sard property") which is intimately related to the zero set of the vector field.

1.2. Renormalized solutions and regular Lagrangian flows

In the last twenty years a great interest has grown about the study of the (multidimensional) ordinary differential equation

$$\begin{cases} \dot\gamma(t) = f\big(t, \gamma(t)\big), \\ \gamma(0) = x_0 \end{cases} \tag{1.2}$$

and of the (closely related) continuity equation, for the unknown $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$

$$\begin{cases} \partial_t u(t,x) + \operatorname{div}\big(f(t,x)u(t,x)\big) = 0, \\ u(0,x) = \bar u(x), \end{cases} \tag{1.3}$$

when the vector field $f(t,x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is not Lipschitz, but just in some weak differentiability class. This study was motivated by applications to many nonlinear PDEs of the mathematical physics (Boltzmann equation, Vlasov–Poisson equation, conservation laws ...), in which nonsmooth velocity fields appear in a natural way, and "bad" behaviors of the involved quantities are related to relevant physical phenomena (shock waves for instance).

The two milestones in this story are the papers by DiPerna and Lions [12] and by Ambrosio [3], in which the cases of Sobolev and *BV* space regularity are respectively considered, in both cases under boundedness assumptions on the distributional spatial divergence $\operatorname{div} f(t, \cdot)$ of the velocity field, and assuming suitable growth conditions on f.

The strategy in both papers exploits the concept of renormalized solutions for the continuity equation (1.3). Showing the renormalization property for a given vector field gives uniqueness for the Cauchy problem (1.3) in the class of (bounded) weak solutions, and from this by now standard approach gives existence and uniqueness for a suitable concept of solution to the ordinary differential equation (1.2), the one of regular Lagrangian flow. The regular Lagrangian flow is the "good" notion of solution in this low regularity context: roughly speaking, among the many nonunique ODE flows, we pick the flow which does not concentrate trajectories (this flow also enjoys suitable stability properties). We shall enter in more details of this theory in Section 3.1. An alternative approach to the theory of regular Lagrangian flows, based on some quantitative estimates along the flow itself, is provided in [10] and gives well posedness for vector fields belonging to the Sobolev space $W^{1,p}$ with $p > 1$. For a detailed exposition of all these results we refer to [4,5,9].

Going back to the square root case $f(x) = \sqrt{|x|}$, we realize that $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}) \cap L^{\infty}_{\mathrm{loc}}(\mathbb{R})$ for all $1 \leqslant p < 2$. However, the above described theory of flows does not cover this particularly simple case. Indeed, (one-sided at least) boundedness of the divergence of the vector field is essential in the proofs of [12] and [3], while in the square root case we just have $\operatorname{div} f(x) = \operatorname{sgn} x/(2\sqrt{|x|}\,)$. A refinement of the condition on the divergence (particularly suitable, for instance, in certain applications to conservation laws) is based on the notion of nearly incompressible vector fields: those vector fields for which there exists a function $\rho(t,x)$, with $1/C \leqslant \rho \leqslant C$, such that

$$\partial_t \rho + \operatorname{div}(f\rho) = 0 \tag{1.4}$$

(see [11] for a detailed exposition of the theory of nearly incompressible vector fields). However, an easy computation shows that, in the case of the square root, the best possible summability we can have on a compressibility element ρ satisfying (1.4) is $L^{\infty}([0,T]; L^{p}_{\mathrm{loc}}(\mathbb{R}))$ for all $p < 2$.

1.3. *Main results of the paper*

In this paper we show how it is possible to treat the case of the square root, and of more general one-dimensional continuous vector fields, within this theory. Related questions are raised by Ambrosio in Example 1.1 of [4], Perthame in Section 6.2 of [17], and Jabin in Section 2.2 of [15]. As we have just remarked it is not clear whether this example fits in the functional framework of the theory of renormalized solutions. For this reason, the question of the well posedness in this framework is not at all trivial.

We show uniqueness for the continuity equation (1.3), in the natural class of solutions $u \in L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$. Our proof does not rely on the renormalization technique, but exploits a reparametrization argument. The main idea consists of two pieces:

- Out of the zeros of f pointwise uniqueness of the trajectories always holds, even without regularity of f, as remarked at the end of Section 1.1;
- Close to a zero of the vector field the continuity equation is rewritten in a new space coordinate, explicitly constructed using a trajectory of the vector field. This allows to "stretch" the vector field in such a way that the singularity is ruled out: in the new variables we simply have a motion with constant unit speed.

A technical but crucial lemma from [1] enables to show that this reparametrized formulation is indeed equivalent to the original one. The only difficulty in the proof is to understand how to deal with the "boundary value problems" that naturally show up after this reparametrization (this issue is responsible for the technical assumption (A3) presented in the following, see also Remark 2.3 for further comments). In a certain sense, this reparametrization argument can be alternatively viewed as a "separation of variables" at the PDE level, or as an implementation of the usual theory of characteristics out of the smooth context. Section 2 will be devoted to this proof.

Once uniqueness for the continuity equation has been obtained, exploiting the abstract theory previously depicted we are able to deduce uniqueness for the regular Lagrangian flow, in the natural class of L^1_{loc} densities. We also prove existence of such flow, and of solutions to the continuity equation, under a further assumption on the vector field (which is necessary in order to avoid the presence of concentrations). We shall present this topic in Section 3.

The basic assumptions we make on the vector field $f:\mathbb{R}\to\mathbb{R}$ are the following:

(A1) f is continuous and nonnegative;
(A2) There is no trajectory going to $\pm\infty$ in finite time, i.e. all solutions γ of $\dot\gamma(t) = f(\gamma(t))$ belong to $L^\infty_{\mathrm{loc}}(\mathbb{R})$;
(A3) The set

$$\mathcal{Z} = \left\{x \in \mathbb{R}:\ f(x) = 0\right\}$$

consists of a finite union of points and closed segments. (We mean that unbounded intervals of the form $(-\infty, b]$ and $[a, +\infty)$ are closed.)

From (A3) it follows that the complement of $\mathcal{Z}$ can be written as a finite union of open intervals, that is

$$\mathbb{R} \setminus \mathcal{Z} = \bigcup_{j=1}^{N} (a_j, b_j),$$

where we choose these open intervals in such a way that $b_j \leqslant a_{j+1}$ for all $j = 1, \dots, N-1$, and possibly $a_1 = -\infty$ or $b_N = +\infty$.

We remark again that no assumptions of regularity or on the near incompressibility (as in (1.4)) of f are made: this makes our result different from those in [16,18]. Some related results are obtained in [8]: in particular, uniqueness is shown in the class of bounded solutions (or in the class of nonnegative solutions), although in general existence is missing in these two classes.

Notice that assumption (A1) is essential to the description we want to make: as in the square root example, we are interested in understanding the occurrence of stoppings of trajectories in flows for which the velocity is always nonnegative. The fact that the vector field is autonomous guarantees that the geometry of our problem (i.e., the intervals (a_j, b_j) and the possible existence of nontrivial trajectories exiting from the points a_j) is independent of time. The continuity of f is appropriate in order to have a well-defined closed set of zeros $\mathcal{Z}$; moreover, thanks to Peano theorem the continuity guarantees the existence of Lipschitz trajectories needed to carry out the reparametrization argument when we deal with nonuniqueness points. We postpone to Remark 4.4 some observations regarding the case in which f changes sign.

We remark that assumption (A2) is satisfied for instance if there exists a constant C such that $f(x) \leqslant C(1 + |x|)$ for all $x \in \mathbb{R}$. Blow ups of the trajectories for $t \to -\infty$ can lead to nonuniqueness for the continuity equation (see Remark 2.6), while blow ups of the trajectories for $t \to +\infty$ imply lack of existence for the regular Lagrangian flow.

Assumption (A3) can be slightly relaxed, as observed in Remark 2.4: it is enough to ask that the set of the zeros of f from which a nontrivial trajectory can start consists just of a finite number of points. For sake of expository simplicity we write our proof in the less general but more transparent case in which (A3) is assumed, and only after we motivate the possibility of weakening it.

Notice that assumptions (A1)–(A3) are (obviously) satisfied for all power-like vector fields $f(x) = |x|^\alpha$ with $0 < \alpha < 1$. As these vector fields are Hölder continuous, one might wonder whether a "functional" proof of the uniqueness could be performed. This seems to be forbidden by the counterexamples in [1] and [2], in which nonuniqueness examples for (two-dimensional) vector fields belonging to all Sobolev classes $W^{s,p}$ with $0 < s < 1$ and $1 \leqslant p \leqslant \infty$ are constructed. It is worth noticing that, on the contrary, it has been recently shown in [7,13] that the stochastic counterpart of this problem, in which a multiplicative stochastic perturbation of Brownian type is considered, has uniqueness of solutions.

2. Uniqueness for the continuity equation

In this section we prove the uniqueness of the solution $u \in L^1_{\mathrm{loc}}([0, T] \times \mathbb{R})$ to the Cauchy problem for the one-dimensional continuity equation

$$\begin{cases} \partial_t u(t,x) + \partial_x\big(f(x)u(t,x)\big) = 0, \\ u(0,x) = \bar{u}(x), \end{cases} \tag{2.1}$$

where the initial datum $\bar{u} \in L^1_{\mathrm{loc}}(\mathbb{R})$ and the vector field $f : \mathbb{R} \to \mathbb{R}$ satisfies the assumptions (A1)–(A3) made in the Introduction. The weak formulation of (2.1) reads as follows

$$\int_0^T \int_{\mathbb{R}} u(t,x)\big[\partial_t\varphi(t,x) + f(x)\partial_x\varphi(t,x)\big]\,dx\,dt + \int_{\mathbb{R}} \bar{u}(x)\varphi(0,x)\,dx = 0 \quad \forall \varphi \in \mathrm{Lip}_c\big([0,T) \times \mathbb{R}\big). \tag{2.2}$$

Let us remark that we consider the continuity equation in (2.1) and not the closely related transport equation $\partial_t u + f \cdot \partial_x u = 0$ since in general the product $f \cdot \partial_x u$ is not defined as a distribution, as we have no assumptions on the local summability of $\operatorname{div} f = \partial_x f$, or on the existence of a compressibility element ρ as in (1.4). In any case, the interpretation of the continuity equation is the suitable one when dealing with flows, as it will be clear from the abstract theory of Section 3.1. (However, notice that in principle, for the particular case of the square root, we have $\partial_x\sqrt{|x|} \in L^1_{\mathrm{loc}}(\mathbb{R})$, thus the usual distributional meaning could be given to the product $f \cdot \partial_x u$.)

The main result of this paper is the following theorem regarding the uniqueness for the Cauchy problem for the continuity equation (2.1).

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy assumptions* (A1)–(A3) *and let $\bar{u} \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then there exists at most one solution $u \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R})$ to the Cauchy problem for the continuity equation* (2.1).

In the course of the proof of the above theorem we will be using a density lemma involving Lipschitz functions, shown in [1]. Let us consider a (bounded or unbounded) open interval $I \subset \mathbb{R}$, and a locally Lipschitz injective curve $\gamma : I \to \mathbb{R}$, such that $\dot{\gamma}(s) \neq 0$ for a.e. $s \in I$. We introduce the following class of Lipschitz functions defined on the interval I:

$$X_c(\gamma) = \big\{\varphi \in \mathrm{Lip}_c(I)\colon\ \varphi = \Phi \circ \gamma,\ \Phi \in \mathrm{Lip}_c(\mathbb{R})\big\}. \tag{2.3}$$

We notice that in general the above class is different from the whole $\mathrm{Lip}_c(I)$: indeed, the curve γ could have non-Lipschitz inverse, think for instance to $\gamma(s) = s^2 \operatorname{sgn} s$. However, the following density lemma holds:

Lemma 2.2. *(See [1].) Let $\alpha \in L^1_{\mathrm{loc}}(I)$. Then the following implication holds*:

$$\int_I \alpha\dot{\varphi}\,dx = 0 \quad \forall \varphi \in X_c(\gamma) \quad \Longrightarrow \quad \int_I \alpha\dot{\varphi}\,dx = 0 \quad \forall \varphi \in \mathrm{Lip}_c(I).$$

Let us simply summarize the idea of the proof. If the thesis were false, then α would be different from a constant in I. This means that we could find $a_1 \neq a_2$ such that $\alpha \simeq a_1$ on an interval (s_1, t_1) and $\alpha \simeq a_2$ on a disjoint interval (s_2, t_2), with $t_1 < s_2$. Then we select $\Phi \in \mathrm{Lip}_c(\mathbb{R})$ such that $\Phi = 1$ on (t_1, s_2) and $\Phi = 0$ on $(-\infty, s_1) \cup (t_2, +\infty)$. Considering $\varphi = \Phi \circ \gamma$, we can compute

$$\int_I \alpha\dot{\varphi}\,dx \simeq \int_{s_1}^{t_1} a_1\dot{\varphi}\,dx + \int_{s_2}^{t_2} a_2\dot{\varphi}\,dx = a_1 - a_2 \neq 0.$$

Thus we have constructed a function $\varphi \in X_c(\gamma)$ contradicting the assumption in the implication in Lemma 2.2. This argument can be made precise as shown in Section 7 of [1], to which we refer for a complete proof of the lemma (which in fact holds even in the case of Lipschitz curves γ with values in $\mathbb{R}^d$).

We now go back to the proof of the main theorem.

Proof of Theorem 2.1. As usual, being the continuity equation linear, it suffices to show that the only solution in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R})$ to the Cauchy problem with $\bar u \equiv 0$ is the trivial one. We recall that the weak formulation of (2.1) on an open interval $I \subset \mathbb{R}$ and with initial datum $\bar u \equiv 0$, obtained by particularizing (2.2), is given by

$$\int_0^T \int_{\mathbb{R}} u(t,x)\big[\partial_t \varphi(t,x) + f(x)\partial_x \varphi(t,x)\big]\,dx\,dt = 0 \quad \forall \varphi \in \mathrm{Lip}_c\big([0,T)\times I\big). \tag{2.4}$$

Step 1. The solution vanishes for a.e. $x \in \mathcal{Z}$. We use the weak formulation (2.4) with $\varphi(t,x) = \varphi^1(t)\varphi^2(x)$, where $\varphi^1 \in \mathrm{Lip}_c([0,T))$ and $\varphi^2 \in \mathrm{Lip}_c(\mathrm{Int}(\mathcal{Z}))$, where we have denoted by $\mathrm{Int}(\mathcal{Z})$ the (topological) interior of the set $\mathcal{Z}$. This gives

$$\int_{\mathbb{R}} \left[\int_0^T u(t,x)\partial_t \varphi^1(t)\,dt \right] \varphi^2(x)\,dx = 0.$$

By the arbitrariness of φ^1 and φ^2 we conclude that $u(t,x) = 0$ for a.e. $t \in [0,T]$ and for a.e. $x \in \mathrm{Int}(\mathcal{Z})$, and by (A3) this is also true for a.e. $x \in \mathcal{Z}$.

Step 2. Propagation of the uniqueness. We show that for all $j = 1,2,\dots,N$ the following implication holds:

$$\left.\begin{array}{l} u(t,x) = 0, \\ \text{for a.e. } t \in [0,T], \\ \text{and for a.e. } x \in (-\infty, a_j) \end{array}\right\} \implies \begin{cases} u(t,x) = 0, \\ \text{for a.e. } t \in [0,T], \\ \text{and for a.e. } x \in (a_j, b_j). \end{cases} \tag{2.5}$$

We first assume that $a_1 \neq -\infty$. We fix j and we have to consider two cases.

Case 1. *There exists a trajectory of f exiting from a_j.* In this first case we assume that there exists $\gamma_j : [0,\sigma_j) \to \mathbb{R}$ (with possibly $\sigma_j = +\infty$) which solves the ODE

$$\begin{cases} \dot\gamma_j(s) = f\big(\gamma_j(s)\big) & \text{for } s \in (0,\sigma_j), \\ \gamma_j(0) = a_j \end{cases}$$

and satisfies

$$\gamma_j\big((0,\sigma_j)\big) = (a_j, b_j).$$

Notice that these conditions, together with the continuity and the strict positivity of f in (a_j, b_j), imply that actually

$$\gamma_j : (0,\sigma_j) \to (a_j, b_j)$$

is C^1 and bijective, with $\dot\gamma_j(s) \neq 0$ for all $s \in (0,\sigma_j)$.

We set

$$\hat{f}_j(x) = \begin{cases} 1 & \text{for } x \in (-\infty, a_j), \\ f(x) & \text{for } x \in [a_j, b_j) \end{cases}$$

and we notice that, by the hypothesis in the implication (2.5), u is also a weak solution to the equation

$$\partial_t u + \partial_x\big(\hat{f}_j(x)u(t,x)\big) = 0 \quad \text{in } (0,T) \times (-\infty, b_j). \tag{2.6}$$

Moreover, the Lipschitz curve $\hat{\gamma}_j : (-\infty, \sigma_j) \to \mathbb{R}$ defined by

$$\hat{\gamma}_j(s) = \begin{cases} s + a_j & \text{for } s < 0, \\ \gamma_j(s) & \text{for } s \geqslant 0 \end{cases}$$

is a trajectory of the vector field $\hat{f}_j$, with $\hat{\gamma}_j(0) = a_j$, being Lipschitz with $\dot{\hat{\gamma}}_j(s) \neq 0$ for a.e. $s \in (-\infty, \sigma_j)$, and a bijection between $(-\infty, \sigma_j)$ and $(-\infty, b_j)$.

We start from the weak formulation of (2.6) with test functions of the form $\varphi(t,x) = \varphi^1(t)\varphi^2(x)$ with $\varphi^1 \in \mathrm{Lip}_c([0,T))$ and $\varphi^2 \in \mathrm{Lip}_c((-\infty, b_j))$. Remember that such products are dense in the class of all test functions. We obtain

$$\int_0^T \int_{-\infty}^{b_j} u(t,x)\big[\big(\partial_t\varphi^1(t)\big)\varphi^2(x) + \hat{f}_j(x)\varphi^1(t)\big(\partial_x\varphi^2(x)\big)\big]\,dx\,dt = 0. \tag{2.7}$$

We now change variable according to $x = \hat{\gamma}_j(s)$. After some computations, in which we use at various times the identity $\dot{\hat{\gamma}}_j(s) = \hat{f}_j(\hat{\gamma}_j(s))$, we obtain

$$\int_0^T \int_{-\infty}^{\sigma_j} \hat{u}_j(t,s)\big[\big(\partial_t\varphi^1(t)\big)\hat{\varphi}_j^2(s) + \varphi^1(t)\big(\partial_s\hat{\varphi}_j^2(s)\big)\big]\,ds\,dt = 0, \tag{2.8}$$

where we have set

$$\hat{u}_j(t,s) = u\big(t, \hat{\gamma}_j(s)\big)\hat{f}_j\big(\hat{\gamma}_j(s)\big) \quad \text{and} \quad \hat{\varphi}_j^2(s) = \varphi^2\big(\hat{\gamma}_j(s)\big). \tag{2.9}$$

Notice that Eq. (2.8) is not exactly a distributional equation in $(0,T) \times (-\infty, \sigma_j)$, since the only test functions in space that are allowed are the $\hat{\varphi}_j^2$ of the above form. For this reason we need to use the density result in Lemma 2.2. First we use Fubini theorem and an integration by parts to rewrite (2.9) as

$$\int_{-\infty}^{\sigma_j} \left[\int_0^T \hat{u}_j(t,s)\varphi^1(t)\,dt - \int_{-\infty}^{s}\int_0^T \hat{u}_j(t,r)\partial_t\varphi^1(t)\,dt\,dr\right]\partial_s\hat{\varphi}_j^2(s)\,ds = 0. \tag{2.10}$$

We are precisely in the setting of Lemma 2.2, as the expression between square brackets in (2.10) is a function in $L^1_{\mathrm{loc}}((-\infty, \sigma_j))$. We deduce from the lemma that

$$\int_{-\infty}^{\sigma_j} \left[\int_0^T \hat{u}_j(t,s)\varphi^1(t)\,dt - \int_{-\infty}^{s}\int_0^T \hat{u}_j(t,r)\partial_t\varphi^1(t)\,dt\,dr\right]\partial_s\psi(s)\,ds = 0 \tag{2.11}$$

holds for all $\psi \in \mathrm{Lip}_c((-\infty, \sigma_j))$. Now, reversing the previous computations, we realize that (2.11) is the weak form of the continuity equation

$$\partial_t \hat{u}_j + \partial_s \hat{u}_j = 0$$

for the unknown $\hat{u}_j$ in the domain $(0, T) \times (-\infty, \sigma_j)$, with initial data $\hat{u}_j(0, \cdot) \equiv 0$ in $(-\infty, \sigma_j)$. Thus we have $\hat{u}_j = 0$ a.e. in $(0, T) \times (-\infty, \sigma_j)$, and from (2.9) (and the properties of $\hat{\gamma}_j$ and $\hat{f}_j$) we also have $u = 0$ a.e. in $(0, T) \times (-\infty, b_j)$. We have shown implication (2.5) in this first case.

Case 2. *There exists no trajectory of f exiting from a_j.* In this case we can find $\gamma_j : (-\infty, \sigma_j) \to \mathbb{R}$ (with possibly $\sigma_j = +\infty$) which solves the ODE

$$\begin{cases} \dot{\gamma}_j(s) = f\big(\gamma_j(s)\big) & \text{for } s \in (-\infty, \sigma_j), \\ \lim\limits_{s \to -\infty} \gamma_j(s) = a_j \end{cases}$$

and satisfies

$$\gamma_j\big((-\infty, \sigma_j)\big) = (a_j, b_j).$$

Notice that these conditions, together with the continuity and the strict positivity of f in (a_j, b_j), imply that actually

$$\gamma_j : (-\infty, \sigma_j) \to (a_j, b_j)$$

is C^1 and bijective, with $\dot{\gamma}_j(s) \neq 0$ for all $s \in (-\infty, \sigma_j)$.

We choose $\varphi(t, x) = \varphi^1(t)\varphi^2(x)$ in (2.4), with $\varphi^1 \in \mathrm{Lip}_c([0, T))$ and $\varphi^2 \in \mathrm{Lip}_c((a_j, b_j))$. Changing variable according to $x = \gamma_j(s)$ we deduce

$$\int_0^T \int_{-\infty}^{\sigma_j} \hat{u}_j(t, s)\big[\big(\partial_t \varphi^1(t)\big)\hat{\varphi}_j^2(s) + \varphi^1(t)\big(\partial_s \hat{\varphi}_j^2(s)\big)\big]\, ds\, dt = 0 \tag{2.12}$$

where we have set

$$\hat{u}_j(t, s) = u\big(t, \gamma_j(s)\big) f_j\big(\gamma_j(s)\big) \quad \text{and} \quad \hat{\varphi}_j^2(s) = \varphi^2\big(\gamma_j(s)\big). \tag{2.13}$$

We conclude by applying exactly the same argument as in Case 1. Notice that in this second case the validity of the thesis in implication (2.5) does not depend on the validity of the assumption (see Remark 2.3). Moreover, the use of Lemma 2.2 is not strictly necessary in this case, since γ_j has a Lipschitz inverse when restricted to the set $\gamma_j^{-1}(\operatorname{spt} \varphi^2)$, for every fixed $\varphi^2 \in \mathrm{Lip}_c((a_j, b_j))$.

In the case $a_1 = -\infty$ the only change is in the proof of the implication (2.5) for $j = 1$. But thanks to assumption (A2) we can find an integral curve of the vector field $\gamma_1 : (-\infty, \sigma_1) \to (-\infty, b_1)$, with possibly $\sigma_1 = +\infty$, which is C^1 and bijective, with $\dot{\gamma}_1(s) \neq 0$ for all s. Using γ_1, we can argue exactly as in Case 2 above. The necessity of assumption (A2) is clarified in Remark 2.6.

Step 3. Conclusion of the proof. An immediate induction argument based on Steps 1 and 2 gives that the solution vanishes a.e. in the intervals (a_j, b_j). This is sufficient to conclude, recalling again the result of Step 1. □

Remark 2.3. We observe that there is a striking difference between the two cases considered in Step 2 in the proof of Theorem 2.1.

In the first case, since there exists a trajectory exiting (in finite time) from a_j, the solution in (a_j, b_j) "sees" the values of the solution in a left neighborhood of a_j. The need of considering the equation on the space domain $(-\infty, b_j)$ and the modification $\hat{f}_j$ of the original vector field f precisely comes from the need of handing in some way the "boundary condition" at a_j (note that characteristic curves enter the domain (a_j, b_j) at a_j). The issue of defining a trace of $u(t,x)$ at $x = a_j$ is in principle extremely delicate, since the solution u enjoys no regularity besides (local) summability. The reparametrization inside the interval (a_j, b_j) can be carried out even in the absence of assumption (A3), but in that case it would be unclear how to prescribe a "correct" left boundary datum.

On the contrary, in the second case no trajectory exits from a_j, so heuristically no information on the behavior of u on $(-\infty, a_j)$ is needed in order to understand the behavior of u on (a_j, b_j). At a technical level, this is due to the fact that the domain of γ_j is unbounded to the left, so that no left boundary appears.

Remark 2.4. Consider the set $\hat{\mathcal{Z}}$ consisting of those $z \in \mathcal{Z}$ such that the ODE

$$\begin{cases} \dot{\gamma}(s) = f\big(\gamma(s)\big), \\ \gamma(0) = z \end{cases}$$

for $s \geqslant 0$ has a nontrivial solution $\gamma(s) \not\equiv z$. We want to show that assumption (A3) can be replaced by the following weaker one:

(A3′) The set $\hat{\mathcal{Z}}$ consists of a finite number of points.

The proof is a simple adaptation of the one of Theorem 2.1. Let us notice, however, that while assumption (A3) can be checked via a direct inspection of the zero set of f, assumption (A3′) requires the study of the associated ODE in order to be verified.

We write $\hat{\mathcal{Z}} = \{z_1, z_2, \dots, z_M\}$, where the z_j are in increasing order. For every $z \in \mathcal{Z} \setminus \hat{\mathcal{Z}}$ let $j(z)$ be the least integer such that $z_{j(z)} > z$.

Arguing as in Case 2 of Step 2 (and possibly in Step 1) of the proof of Theorem 2.1 one first shows that $u(t,x) \equiv 0$ for a.e. $t \in [0,T]$ and for a.e. x belonging to

$$(-\infty, z_1) \cup \bigcup_{z \in \mathcal{Z} \setminus \hat{\mathcal{Z}}} (z, z_{j(z)}).$$

After, it remains to discuss what happens in the complement of the above set, which consists of the union of intervals of the form (z_j, w_j), where $w_j \in \mathcal{Z}$ and $w_j \leqslant z_{j+1}$, for which there holds $(z_j, w_j) \cap \mathcal{Z} = \emptyset$. Along the same line of Case 1 of the proof of the above theorem, it can be shown by induction on j that $u(t,x)$ vanishes for a.e. x belonging to the union of such intervals and for a.e. $t \in [0,T]$, concluding the proof.

Remark 2.5. It would be interesting to understand what happens if (A3) is completely removed, i.e., when no assumptions are made on the zero set $\mathcal{Z}$. Let us notice that our proof cannot handle cases in which, for instance, it happens that

$$\hat{\mathcal{Z}} = \{0\} \cup \left\{ -\frac{1}{n} \colon n = 1, 2, \dots \right\}$$

and

$$\mathcal{Z} \setminus \hat{\mathcal{Z}} = \left\{ -\frac{1}{n} - \frac{1}{n^2} \colon n = 1, 2, \ldots \right\},$$

where $\hat{\mathcal{Z}}$ is as in Remark 2.4.

A related example is presented in Section IV.1 of [12]. However, for this example all flows but one create concentration of trajectories, as noticed in Section 5.3.1 of [9], so the regular Lagrangian flow is nevertheless unique.

Remark 2.6. If assumption (A2) is violated we can have nonuniqueness for the continuity equation (2.1), even for smooth vector fields. For instance, the function defined by

$$u(t,x) = \begin{cases} 1/x^2 & \text{if } 0 < -1/x < t, \\ 0 & \text{otherwise} \end{cases}$$

belongs to $L^\infty([0,T]; L^1(\mathbb{R})) \cap L^\infty([0,T] \times \mathbb{R})$, solves the continuity equation $\partial_t u + \partial_x(x^2 u) = 0$ and attains value zero at initial time.

The proof of Theorem 2.1 breaks down in the last passage of Step 2, when the case $a_1 = -\infty$ is considered. Indeed, the domain of the parametrization trajectory γ_1 would be bounded to the left, but this would allow to prescribe arbitrary left boundary data. In some sense, some mass coming from infinity can instantaneously appear, giving rise to a nonzero solution.

3. Regular Lagrangian flows

We now want to deal with the ODE side of the problem. We start by introducing the notion of regular Lagrangian flow with L^1_{loc} densities.

Definition 3.1 *(L^1_{loc}-regular Lagrangian flow).* Let us fix a vector field $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$. We say that a map $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is an L^1_{loc}-regular Lagrangian flow relative to f if the two following conditions are satisfied:

(i) For a.e. $x \in \mathbb{R}^d$ the map $t \mapsto X(t,x)$ is an absolutely continuous integral solution of the ordinary differential equation $\dot{\gamma}(t) = f(t, \gamma(t))$;
(ii) For a.e. $t \in [0,T]$ there holds $X(t,\cdot)_\# \mathscr{L}^d \ll \mathscr{L}^d$.

Observe that property (ii) in Definition 3.1 forbids (heuristically at least) the phenomenon of concentration of trajectories presented (for the square root case) in the Introduction. This gives some hope that solutions in the sense of L^1_{loc}-regular Lagrangian flows should be unique: heuristically, we disregard those solutions that have stoppings, and thus concentrations.

Remark 3.2. Condition (ii) in Definition 3.1 has also the advantage of making the notion of L^1_{loc}-regular Lagrangian flow independent of the choice of the pointwise value of f in a negligible set. More precisely, if $f = \tilde{f}$ a.e. in $[0,T] \times \mathbb{R}^d$, we have that X is an L^1_{loc}-regular Lagrangian flow associated to f if and only if it is an L^1_{loc}-regular Lagrangian flow associated to $\tilde{f}$.

3.1. Abstract theory of regular Lagrangian flows and uniqueness

We now briefly recall some points of the abstract theory of regular Lagrangian flows in $\mathbb{R}^d$ as presented in [4,5]. Using this abstract machinery, the uniqueness of the L^1_{loc}-regular Lagrangian flow for vector fields $f : \mathbb{R} \to \mathbb{R}$ satisfying assumptions (A1)–(A3) (or (A1)–(A3$'$)) will be an immediate consequence of Theorem 2.1.

Let us fix a convex class $\mathcal{L}$ of measure-valued solutions μ_t of the continuity equation with some (fixed) Borel vector field $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, and assume that $\mathcal{L}$ satisfies the implication

$$0 \leqslant \mu'_t \leqslant \mu_t \in \mathcal{L} \quad \Longrightarrow \quad \mu'_t \in \mathcal{L}.$$

We say that a Borel map $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is an $\mathcal{L}$-regular Lagrangian flow associated to f and starting from $\bar{\mu} \in \mathcal{L}$ if the two following conditions are satisfied:

(i) For a.e. $x \in \mathbb{R}^d$ the map $t \mapsto X(t,x)$ is an absolutely continuous integral solution of the ordinary differential equation $\dot{\gamma}(t) = f(t, \gamma(t))$;
(ii) For a.e. $t \in [0,T]$ there holds $X(t,\cdot)_\# \bar{\mu} \in \mathcal{L}$.

Then the following implication holds. If the continuity equation with vector field f has the uniqueness property in the class $\mathcal{L}$, then for all $\bar{\mu} \in \mathcal{L}$ the $\mathcal{L}$-regular Lagrangian flow starting from $\bar{\mu}$ is unique. We stress that with uniqueness property for the continuity equation in $\mathcal{L}$ we mean that, if μ_t and μ'_t are solutions of the continuity equation belonging to $\mathcal{L}$ and having the same value for $t = 0$, then we must have $\mu_t = \mu'_t$ as measures for a.e. $t \in [0,T]$. We understand that we systematically choose the representative of the solution μ_t in such a way that $[0,T] \ni t \mapsto \mu_t \in \mathcal{M}(\mathbb{R}^d)$ is weakly-$*$ continuous (see Lemma 8.1.2 in [6]). In particular this gives a sense to $\mu_t \in \mathcal{M}(\mathbb{R}^d)$ for all values of $t \in [0,T]$.

The proof of the above implication strongly relies on the so-called superposition principle (for which we refer for instance to Section 3 in [4]), which roughly speaking asserts that every positive measure-valued solution to the continuity equation can be represented as a weighted push forward of the initial datum along a superposition of the (possibly nonunique) solutions of the ODE.

In the usual case in which boundedness of the divergence $\operatorname{div} f(t,\cdot)$ is assumed, the natural class $\mathcal{L}$ consists of positive functions in $L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$: indeed, the bound on the divergence ensures a maximum principle, that in turn implies existence in this class. This does not hold in our context: even in the case when nonsingular solutions exist, they are expected to be unbounded, so we are forced to consider solutions which just belong to L^1_{loc}.

3.2. Uniqueness

Applying the abstract result we have just described to our particular setting, in which we take as $\mathcal{L}$ the class of positive functions in $L^1_{\mathrm{loc}}(\mathbb{R})$ and $\bar{\mu} = \mathscr{L}^1$, the following theorem immediately follows from Theorem 2.1.

Theorem 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy assumptions* (A1)–(A3) (*or* (A1)–(A3′)). *Then there exists at most one L^1_{loc}-regular Lagrangian flow associated to f.*

3.3. Existence and stability

We want now to face the problem of the existence of an L^1_{loc}-regular Lagrangian flow associated to f and of solutions $u \in L^1_{\mathrm{loc}}([0,T] \times \mathbb{R})$ to the Cauchy problem for the continuity equation (2.1) when the initial datum $\bar{u} \in L^1_{\mathrm{loc}}(\mathbb{R})$.

In general, there is no existence in L^1_{loc} under the sole assumptions (A1)–(A3). Indeed, consider for instance

$$f(x) = \begin{cases} \sqrt{|x|} & \text{for } x < 0, \\ x & \text{for } x \geqslant 0. \end{cases}$$

It is clear that we have pointwise uniqueness for the ODE with vector field f for every initial point $x_0 \in \mathbb{R}$, but trajectories starting from every $x_0 < 0$ reach the origin in finite time and stay there indefinitely, so that concentrations must take place.

We want to show that this phenomenon is essentially the only possible obstruction to the existence. We shall assume the following additional condition, which says that, if it is possible to reach a zero in finite time, then it is also possible to exit from it: thus, the flow is not forced to create concentrations.

(A4) For every $z \in \mathcal{Z}$ the following implication holds: if there exists a nonconstant trajectory $\gamma_{\mathrm{in}} : (-\sigma, 0] \to \mathbb{R}$ (with $\sigma > 0$) for which

$$\begin{cases} \dot\gamma(s) = f\big(\gamma(s)\big), \\ \gamma(0) = z, \end{cases} \tag{3.1}$$

then there exists a nonconstant trajectory $\gamma_{\mathrm{out}} : [0, \tau) \to \mathbb{R}$ (with $\tau > 0$) which also solves (3.1).

Theorem 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy assumptions* (A1)–(A4). *Then there exists an L^1_{loc}-regular Lagrangian flow associated to f. Moreover, for every $\bar u \in L^1_{\mathrm{loc}}(\mathbb{R})$, there exists a solution $u \in L^1_{\mathrm{loc}}([0, T] \times \mathbb{R})$ to the Cauchy problem for the continuity equation* (2.1).

The following corollary follows by combining Theorems 2.1 and 3.4. Notice that both the theorem and the corollary hold if assumption (A3) is replaced with assumption (A3′).

Corollary 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfy assumptions* (A1)–(A4). *Then there exists a unique L^1_{loc}-regular Lagrangian flow associated to f. Moreover, for every $\bar u \in L^1_{\mathrm{loc}}(\mathbb{R})$, there exists a unique solution $u \in L^1_{\mathrm{loc}}([0, T] \times \mathbb{R})$ to the Cauchy problem for the continuity equation* (2.1).

Proof of Theorem 3.4. Consider the subset $\bar{\mathcal{Z}}$ of $\mathcal{Z}$ consisting of those $z \in \mathcal{Z}$ for which there exists no trajectory of f entering in z in finite time. From assumption (A4) it follows that $\bar{\mathcal{Z}}$ consists of all the nontrivial closed segments in $\mathcal{Z}$, while some of the isolated points of $\mathcal{Z}$ may fail to belong to $\bar{\mathcal{Z}}$ (precisely those points in which characteristics can enter and immediately exit).

Moreover, the open set $\mathbb{R} \setminus \bar{\mathcal{Z}}$ is the union of a finite number of open intervals J_i, and for every such interval there exists a trajectory γ_i of f such that

$$\gamma_i : (\sigma_i, +\infty) \to J_i$$

is C^1 and bijective, with $\dot\gamma_i(s) \neq 0$ for a.e. $s \in (\sigma_i, +\infty)$. Notice in particular that all the domains of the parametrizations γ_i are unbounded to the right, while it can happen that either $\sigma_i = -\infty$ or $\sigma_i \in \mathbb{R}$.

Given $\bar u \in L^1_{\mathrm{loc}}(\mathbb{R})$, we want to construct a solution in $L^1_{\mathrm{loc}}([0, T] \times \mathbb{R})$ of the continuity equation (2.1) with initial data $\bar u$. Following the computations in the proof of Theorem 2.1 it is easy to derive the expression

$$u(t, x) = \bar u(x) \mathbf{1}_{\bar{\mathcal{Z}}}(x) + \sum_i \rho_i(t, x) \mathbf{1}_{J_i}(x),$$

where the densities ρ_i are explicitly given by the formula

$$\rho_i(t, x) = \frac{f(\gamma_i(\gamma_i^{-1}(x) - t))}{f(x)} \bar u\big(\gamma_i\big(\gamma_i^{-1}(x) - t\big)\big) \mathbf{1}_{(\sigma_i + t, +\infty)}\big(\gamma_i^{-1}(x)\big).$$

An explicit computation shows that $u \in L^1_{\mathrm{loc}}([0, T] \times \mathbb{R})$ and is a solution of (2.1).

Finally, the existence of an L^1_{loc}-regular Lagrangian flow associated to f follows by the abstract theory of regular Lagrangian flows: simply apply Theorem 4.11 in [5] with $\mathcal{L}$ given by the class of positive functions in $L^1_{\mathrm{loc}}(\mathbb{R})$ and $\bar\mu = \mathscr{L}^1$. $\square$

Let us notice that the L^1_{loc}-regular Lagrangian flow $X(t,x)$ associated to f given by the previous theorem can be explicitly characterized as follows. For all $x \in \bar{\mathcal{Z}}$ we simply set $X(t,x) = x$. For all $x \in \mathbb{R} \setminus \bar{\mathcal{Z}}$ we require that $t \mapsto X(t,x)$ is the unique trajectory of f which is strictly increasing in time. Let us also notice that, if $x \in J_i$, then $X(t,x) \in J_i$ for all $t \in [0,T]$.

Remark 3.6 *(Lack of stability).* We observe that, although under assumptions (A1)–(A4) we have existence and uniqueness of the L^1_{loc}-regular Lagrangian flow, in general stability can be missing. It has been pointed out in Example 1.1 of [4] that, already in the case $f(x) = \sqrt{|x|}$, for every $t_0 \geqslant 0$ it is possible to construct a Lipschitz (or even smooth) approximation of the vector field

$$f_\varepsilon^{t_0} \to f \quad \text{locally uniformly in } \mathbb{R} \text{ as } \varepsilon \to 0$$

in such a way that the (unique) classical flows associated to $f_\varepsilon^{t_0}$ converge to the flow of f constructed by letting each trajectory stop at the origin for precisely a time t_0. Thus the L^1_{loc}-regular Lagrangian flow associated to $f(x) = \sqrt{|x|}$ is not stable with respect to smooth approximations.

This seems to indicate that L^∞ boundedness of the divergence (or at least L^∞ boundedness of the densities of $X(t,\cdot)_\# \mathscr{L}^d$ in Definition 3.1(ii)) is essential in stability theorems for regular Lagrangian flows (see for instance Theorem 6.4.6 in [9]): it is not enough to have existence and uniqueness or the regular Lagrangian flow and merely $\operatorname{div} f(t,\cdot) \ll \mathscr{L}^d$ to have stability.

4. Final remarks

Remark 4.1 *(A direct proof of the uniqueness of the regular Lagrangian flow).* If we were interested just in the uniqueness of the L^1_{loc}-regular Lagrangian flow (and not in the uniqueness of solutions $u \in L^1_{\mathrm{loc}}$ to the continuity equation (2.1)), a much easier proof could have been carried out. Consider a vector field $f : \mathbb{R} \to \mathbb{R}$ satisfying assumptions (A1)–(A3). Then modify f by setting it equal to 1 on the isolated zeros, that is set

$$\tilde{f}(x) = \begin{cases} 1 & \text{if } x \text{ is an isolated point of } \mathcal{Z}, \\ f(x) & \text{otherwise.} \end{cases}$$

Let $X(t,x)$ be an L^1_{loc}-regular Lagrangian flow associated to f. As $\tilde{f} = f$ a.e. in $\mathbb{R}$, Remark 3.2 implies that X is also an L^1_{loc}-regular Lagrangian flow associated to $\tilde{f}$. According to Definition 3.1, for a.e. $x \in \mathbb{R}$ the map $t \mapsto X(t,x)$ is an absolutely continuous integral solution of the ordinary differential equation for both the vector fields f and $\tilde{f}$.

Let us show that, for a.e. $x \in \mathbb{R}$, the trajectory $t \mapsto X(t,x)$ does not stop for a strictly positive amount of time at any isolated zero of f. This claim clearly implies uniqueness for the L^1_{loc}-regular Lagrangian flow associated to f, as $t \mapsto X(t,x)$ is uniquely determined out of the isolated zeros of f, hence $X(t,x)$ would be determined for all $t \in [0,T]$ and for a.e. $x \in \mathbb{R}$.

If the claim were false, we could find initial data x in a set with strictly positive measure, for which the trajectories $t \mapsto X(t,x)$ are integral curves of both f and $\tilde{f}$, and stop for a strictly positive amount of time at some isolated zero of f. However, this is in contrast with the fact that $\tilde{f} = 1$ in those points: this shows the claim.

The above argument essentially relies on the observation that, in the case of the stopping (for a strictly positive time) of the trajectories, pointwise values of the vector field become relevant, but this should not be the case when dealing with regular Lagrangian flows: hence, stoppings are forbidden.

Remark 4.2. From the previous remark it directly follows that uniqueness for the continuity equation holds in the class of *positive* solutions $u \in L^1_{\mathrm{loc}}$ (compare also with similar results in Section 2 of [8]). This is a consequence of the superposition principle (already recalled in Section 3.1). The line of the proof is as follows.

Every positive solution can be rewritten as a superposition of solutions propagated along characteristics, but as we are considering L^1_{loc} solutions we deduce that the involved characteristics are associated to an L^1_{loc}-regular Lagrangian flow. The fact that the L^1_{loc}-regular Lagrangian flow is unique is enough to conclude, as it implies that every L^1_{loc} solution is determined by propagating the initial datum along the characteristics of the L^1_{loc}-regular Lagrangian flow.

The same argument shows the uniqueness of renormalized solutions $u \in L^1_{\mathrm{loc}}$ (for a precise definition of renormalized solution and for the importance of such a notion in this context see [12,3] or the notes [4,5]). Notice that, while in the DiPerna–Lions–Ambrosio theory of renormalized solutions it is needed that the renormalization property holds for the difference of any couple of solutions sharing the same initial datum, here the uniqueness is proved directly inside the class of renormalized solutions.

Remark 4.3. On the contrary it is easy to realize that uniqueness fails in the class of positive measure-valued solutions. Take for instance $f(x) = \sqrt{|x|}$ and consider the flow

$$Y(t,x) = \begin{cases} \text{the trajectory starting from } x \text{ reaches the origin and stays there forever, when } x \leqslant 0, \\ \text{the unique trajectory starting from } x, \text{ when } x > 0. \end{cases}$$

The positive measure $\mu_t = Y(t,\cdot)_\# \mathscr{L}^1$ is a solution to the continuity equation (this easily follows from the theory of superposition solutions). But μ_t is actually singular at $x = 0$ for any $t > 0$, hence it is different from the L^1_{loc} solution constructed in Section 3.3.

Remark 4.4. Finally we want to illustrate some typical situations in which we admit changes of sign of the vector field. First of all, observe that the same argument as in Remark 4.1 shows that a continuous vector field $f : \mathbb{R} \to \mathbb{R}$ (with no sign conditions) which satisfies assumption (A3) has at most one L^1_{loc}-regular Lagrangian flow.

If for instance $f_1(x) = \mathrm{sgn}(x)\sqrt{|x|}$, then there exists a unique L^1_{loc}-regular Lagrangian flow, as there is pointwise uniqueness for the ODE and no concentrations occur; this implies uniqueness of positive solutions (or of renormalized solution), reasoning exactly as in Remark 4.2. However, uniqueness does not hold in the class of L^1_{loc} signed solutions to the continuity equation. Indeed, arguing as in the proof of Theorem 2.1, we can rewrite the continuity equation as two initial boundary value problems, the first one (corresponding to positive values of x) with a left boundary and with positive unit velocity, the second one (corresponding to negative values of x) with a right boundary and with negative unit velocity. It is then clear that, if we impose for the first problem an arbitrary boundary datum $\beta(t)$ at $s = 0$, and for the second one we impose as boundary datum $-\beta(t)$ at $s = 0$, we always get an L^1_{loc} solution. The same kind of construction is presented in Section 6.1 of [13].

On the contrary, taking $f_2(x) = -f_1(x)$ it is readily checked that there exists no L^1_{loc}-regular Lagrangian flow, as all trajectories collapse to the origin in finite time.

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